The Lie Algebra of Local Killing Fields

Richard Atkins

Abstract

We present an algebraic procedure that finds the Lie algebra of the local Killing fields of a smooth metric. In particular, we determine the number of independent local Killing fields about a given point on the manifold. Spaces of constant curvature, locally symmetric spaces and surfaces are also discussed.

1 Introduction

Killing fields describe the infinitesimal isometries of a metric and as such play a significant role in differential geometry and general relativity. In this paper we present an algebraic method that finds the Lie algebra of the local Killing fields of a smooth metric g. In particular, we determine the number of independent local Killing fields of g about any given point. In the section following, we identify the local Killing fields of a metric with local parallel sections of an associated vector bundle W, endowed with a connection ∇. An examination of the form of the curvature of ∇ leads to a characterization of spaces of constant curvature by means of a system of linear equations. In Section 3 we investigate the Lie algebra structure of Killing fields. It is shown that if the Riemann curvature vanishes at some point on the manifold then the Lie algebra of Killing fields is isomorphic to a subalgebra of the Lie algebra of the group of isometries of Euclidean space. Section 4 includes an overview of the procedure developed in [1]. Therein the bundle generated by the local parallel sections of W is found by calculating a derived flag of subsets of W. The number of independent Killing fields of g about a point $x \in M$ is then equal to the dimension of the fibre $\widetilde{W}_x$ over x of the terminal subset of the derived flag. Associated to $\widetilde{W}_x$ is a Lie algebra canonically isomorphic to the Lie algebra $\mathcal{K}_x$ of local Killing fields about x. The method is illustrated by providing a short proof of a classical theorem that gives a necessary condition for a space to be locally symmetric, expressed by the vanishing of a set of quadratic homogeneous polynomials in the curvature. Section 5 considers the derived flag for Riemannian surfaces. We obtain a classification of the Riemannian metrics corresponding to the various possible kinds of Lie algebra $\mathcal{K}_x$.

2 Killing Fields and Constant Curvature

We associate to Killing fields parallel sections of a suitable vector bundle in the manner put forward by Kostant (cf. [9]). The utility of such a framework is two-fold: first, it

permits us to apply algebraic techniques adapted to finding the subbundle generated by local parallel sections. Second, it enables a purely algebraic description of the Lie bracket of two Killing fields, avoiding the explicit appearance of derivatives.

Let g be a metric on a differentiable manifold M of dimension n; g is assumed to be pseudo-Riemannian of signature (p, q) unless otherwise stated. K is a Killing field of g if and only if

$$K_{a;b} + K_{b;a} = 0 \tag{1}$$

where the semi-colon indicates covariant differentiation with respect to the Levi-Civita connection of g. It is straightforward to verify that

$$K_{a;bc} = R_{abc}{}^{d} K_d \tag{2}$$

for Killing fields K, where $R_{abc}{}^{d}$ is the Riemann curvature tensor of g, defined according to

$$A_{c:ba} - A_{c:ab} = R_{abc}{}^{d} A_d$$

The summation convention shall be used throughout.

Let W be the Whitney sum $W := T^*M \oplus \Lambda^2 T^*M$. A local section of W has the form $X = K + L$, where $K = K_a dx^a$ is a local section of T^*M and $L = L_{ab} dx^a \wedge dx^b$ is a local section of $\Lambda^2 T^*M$. Define a connection ∇ on W by

$$\nabla_i X = (K_{a;i} - L_{ai}) dx^a + (L_{ab;i} - R_{abi}{}^{c} K_c) dx^a \wedge dx^b \tag{3}$$

For an open subset $U \subseteq M$, let $\mathcal{K}_U$ denote the local Killing fields $K : U \longrightarrow T^*M$ and let $\mathcal{P}_U$ denote the local parallel sections $X : U \longrightarrow W$; the subscript U shall be omitted when $U = M$. Define the map $\phi_U : \mathcal{K}_U \longrightarrow \mathcal{P}_U$ by

$$\phi_U(K_a) := K_a + K_{a;b} \tag{4}$$

It is clear that the image of ϕ_U does, in fact, lie in $\mathcal{P}_U$. The inverse $\psi_U : \mathcal{P}_U \longrightarrow \mathcal{K}_U$ of ϕ_U is the projection of W onto T^*M: $\psi_U(K_a + L_{ab}) := K_a$. This establishes a

vector space isomorphism

$$\mathcal{K}_U \quad \leftrightarrow \quad \mathcal{P}_U \tag{5}$$

Consider a vector space V with a non-degenerate, symmetric bilinear form h. Let $B = B_{abcd}$ be a covariant 4-tensor on V satisfying the following relations common to a Riemann curvature tensor:

$$B_{abcd} = B_{cdab} = -B_{bacd} = -B_{abdc} \tag{6}$$

and let $T = T_{ab\cdots}$ be an n-tensor on V with $n \geq 2$. The derivation $B \star T$ is the $(n+2)$-tensor defined by

$$B \star T_{abcd\cdots} := B_{sbcd}T^s{}_{a\cdots} + B_{ascd}T^s{}_{b\cdots} + B_{absd}T^s{}_{c\cdots} + B_{abcs}T^s{}_{d\cdots} \tag{7}$$

Indices are raised by h.

Lemma 1 *If V is 2-dimensional then $B \star L = 0$ for all $L \in \Lambda^2 V^*$.*

Proof:

It shall be convenient to work in an orthonormal basis of V in which $h = diag(\eta_1, \eta_2)$, where $\eta_i = \pm 1$. Then $L^i{}_j = \eta_i L_{ij}$. Owing to the symmetries (6), there are effectively two cases to consider.

(i) $a = b = 1$ case:

$$B \star L_{abcd} = \eta_2(B_{21cd} + B_{12cd})L_{21} = 0$$

(ii) $a = c = 1$, $b = d = 2$ case:

$$B \star L_{abcd} = \eta_2 B_{2212}L_{21} + \eta_1 B_{1112}L_{12} + \eta_2 B_{1222}L_{21} + \eta_1 B_{1211}L_{12} = 0$$

q.e.d.

Applying the Bianchi identities, the curvature $F(i,j) := \nabla_{[i}\nabla_{j]}$ of ∇ takes the form

$$F(i,j)(X) = (R_{ijkl;s}K^s + R \star L_{ijkl})dx^k \wedge dx^l \tag{8}$$

where $X = K_a dx^a + L_{ab} dx^a \wedge dx^b$ (cf. [4]). In the sequel, it shall be convenient to view the curvature F as a map $F : W \longrightarrow \Lambda^2 T^*M \otimes W$ given by $w \mapsto F(,)(w)$. F is composed of two pieces: a K-part and an L-part. The K-part provides a description of locally symmetric spaces: g is locally symmetric if and only if $T^*M \subseteq ker\, F$. The L-part, on the other hand, provides a characterization of metrics of constant sectional curvature by means of a system of homogeneous linear equations.

Proposition 2 *Let g be Riemannian and $n \geq 3$. Then M is a space of constant curvature if and only if*

$$R \star L = 0$$

for all $L \in \Lambda^2 T^*M$.

Expressed in terms of indices, M has constant curvature (for $n \geq 3$) if and only if for all $L \in \Lambda^2 T^*M$,

$$R_{sjkl} L^s{}_i + R_{iskl} L^s{}_j + R_{ijsl} L^s{}_k + R_{ijks} L^s{}_l = 0 \tag{9}$$

It is evident from the lemma that the theorem does not hold for $n = 2$.

Proof:

$\Longrightarrow$ If g has constant curvature then $R_{ijkl} = \kappa_0(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$ with respect to an orthonormal frame, where κ_0 is a constant. Substitution of this expression into the left hand side of (9) gives zero for all skew-symmetric $L = L_{ab}$.

$\Longleftarrow$ Suppose that (9) holds for all $L \in \Lambda^2 T^*M$. We shall work in an orthonormal frame $X_1, ..., X_n$ for g; this will allow us to deal with lowered indices throughout: $L^i{}_j = L_{ij}$. Let $i = k, j$ and l be three distinct indices in (9). This gives

$$R_{sjil} L_{si} + R_{isil} L_{sj} + R_{ijsl} L_{si} + R_{ijis} L_{sl} = 0 \tag{10}$$

Put $L_{rs} := \delta_{rl}\delta_{sj} - \delta_{rj}\delta_{sl}$ into (10) to obtain $R_{ijij} = R_{ilil}$. It follows that for any two pairs of distinct indices $i \neq j$ and $a \neq b$, $R_{ijij} = R_{abab}$. Thus

$$R_{ijij} = \kappa(x) \qquad \text{for } i \neq j \tag{11}$$

where κ is some function on M.

Next, let $i = k$ and $j = l$ be two distinct indices in (9). This gives:

$$R_{ijis}L_{sj} + R_{ijsj}L_{si} = 0 \tag{12}$$

Let m be any index distinct from i and j and put $L_{rs} := \delta_{rm}\delta_{sj} - \delta_{rj}\delta_{sm}$ into (12). We obtain

$$R_{ijim} = 0 \qquad \text{for } i, j \text{ and } m \text{ distinct} \tag{13}$$

Consider a pair Y_1, Y_2 of orthonormal vectors in T_xM. If X_1, X_2 span the same plane as Y_1, Y_2 at x then $R(Y_1, Y_2, Y_1, Y_2) = \kappa(x)$, by (11). If Y_1, Y_2 span a plane orthogonal to X_1, X_2 then we may as well suppose $X_3 = Y_1$ and $X_4 = Y_2$, whence $R(Y_1, Y_2, Y_1, Y_2) = \kappa(x)$, from (11) again. The last possibility is that Y_1, Y_2 and X_1, X_2 span planes that intersect through a line, which for the purpose of calculating sectional curvature we may take to be generated by $X_1 = Y_1$, by means of appropriate rotations of the pairs X_1, X_2 and Y_1, Y_2 within the respective planes they span. We may suppose, furthermore, that X_3 is the normalized component of Y_2 orthogonal to X_2; thus $Y_2 = aX_2 + bX_3$, where $a^2 + b^2 = 1$. From (11) and (13) this gives

$$\begin{aligned} R(Y_1, Y_2, Y_1, Y_2) &= R(X_1, aX_2 + bX_3, X_1, aX_2 + bX_3) \\ &= a^2 R_{1212} + b^2 R_{1313} \\ &= \kappa(x) \end{aligned}$$

Therefore g has constant curvature at each point $x \in M$. By Schur's Theorem, g has constant curvature.

q.e.d.

3 The Lie algebra Structure of $\mathcal{K}_U$

Let V be an n-dimensional vector space equipped with a non-degenerate, symmetric bilinear form h, of signature (p, q), and let $B = B_{abcd}$ be a covariant 4-tensor on V

satisfying the usual algebraic relations of a Riemann curvature:

$$\begin{aligned}
B_{abcd} &= -B_{bacd} && (14)\\
B_{abcd} &= -B_{abdc} && (15)\\
B_{abcd} &+ B_{acdb} + B_{adbc} = 0, \quad \text{and afortiori} && (16)\\
B_{abcd} &= B_{cdab}
\end{aligned}$$

By virtue of (14) and (15) we may define a skew-symmetric, bilinear bracket operation on $V^* \oplus \Lambda^2 V^*$ by

$$[K_a + L_{ab}, K'_a + L'_{ab}] := L'_{ab}K^b - L_{ab}K'^b + L'_a{}^c L_{cb} - L_a{}^c L'_{cb} + B_{abcd}K^c K'^d \quad (17)$$

where indices are raised and lowered with h. If a subspace $\mathcal{W}$ of $V^* \oplus \Lambda^2 V^*$ is closed with respect to the bracket and satisfies the Jacobi identity then we denote the associated Lie algebra by $\mathcal{A}(\mathcal{W}, B, h)$.

Lemma 3 *Let $\mathcal{W}$ be a subspace of $V^* \oplus \Lambda^2 V^*$, closed with respect to the bracket operation. The Jacobi identity holds on $\mathcal{W}$ if and only if for all $X = K + L, X' = K' + L'$ and $X'' = K'' + L''$ in $\mathcal{W}$, where $K, K', K'' \in V^*$ and $L, L', L'' \in \Lambda^2 V^*$,*

$$B \star L_{abcd} K'^c K''^d + B \star L'_{abcd} K''^c K^d + B \star L''_{abcd} K^c K'^d = 0 \quad (18)$$

Proof:

Let $K, K', K'' \in V^*$ and $L, L', L'' \in \Lambda^2 V^*$. There are four cases to consider.

(i) $K - K' - K''$ case. We have

$$[K, [K', K'']] = [K, B_{abcd} K'^c K''^d] = B_{abcd} K^b K'^c K''^d$$

Therefore,

$$\begin{aligned}
& [K, [K', K'']] + [K', [K'', K]] + [K'', [K, K']] \\
=\; & (B_{abcd} + B_{acdb} + B_{adbc}) K^b K'^c K''^d \\
=\; & 0 && (19)
\end{aligned}$$

by equation (16).

(ii) $K - K' - L$ case. First,

$$[K, [K', L]] = [K, L_{ab}K'^b] = B_{abcd}K^cL^d{}_sK'^s$$

Also,

$$\begin{aligned}
[L, [K, K']] &= [L, B_{abcd}K^cK'^d] \\
&= B_{ascd}K^cK'^dL^s{}_b - L_a{}^sB_{sbcd}K^cK'^d
\end{aligned}$$

Combining these with (15) and the fact that $L = L_{ab}$ is skew-symmetric, we obtain

$$[K, [K', L]] + [K', [L, K]] + [L, [K, K']] = B \star L_{abcd}K^cK'^d \tag{20}$$

(iii) $K - L - L'$ case. Observe that

$$[K, [L, L']] = [K, L'L - LL'] = L'LK - LL'K$$

and

$$[L, [L', K]] = -[L, L'K] = LL'K$$

Using these equations gives

$$[K, [L, L']] + [L, [L', K]] + [L', [K, L]] = 0 \tag{21}$$

(iv) $L - L' - L''$ case. It is elementary to verify that

$$[L, [L', L'']] + [L', [L'', L]] + [L'', [L, L']] = 0 \tag{22}$$

After applying (19)-(22),

$$[X, [X', X'']] + [X', [X'', X]] + [X'', [X, X']]$$

simplifies to

$$B \star L_{abcd}K'^cK''^d + B \star L'_{abcd}K''^cK^d + B \star L''_{abcd}K^cK'^d$$

q.e.d.

Proposition 4 *(i) If V is 2-dimensional then any subspace $\mathcal{W}$ of $V^* \oplus \Lambda^2 V^*$, closed with respect to the bracket, defines a Lie algebra $\mathcal{A}(\mathcal{W}, B, h)$.*
(ii) If V is arbitrary and $B = 0$ then $V^* \oplus \Lambda^2 V^*$ defines a Lie algebra $\mathcal{A}(V^* \oplus \Lambda^2 V^*, B = 0, h)$.

Proof:
(i) follows from Lemmas 1 and 3. (ii) is immediate.
q.e.d.

For local Killing fields $K, K' \in \mathcal{K}_U$, the Lie bracket $K'' := [K, K']$ is

$$K''_a = L'_{ab} K^b - L_{ab} K'^b \tag{23}$$

where we have written $L_{ab} := K_{a;b}$ and $L'_{ab} := K'_{a;b}$. Furthermore, $L''_{ab} := K''_{a;b}$ is given by

$$L''_{ab} \quad = \quad L'_a{}^c L_{cb} - L_a{}^c L'_{cb} + R_{abcd} K^c K'^d \tag{24}$$

Defining a bracket on $\mathcal{P}_U$ by

$$[K + L, K' + L'] := L'_{ab} K^b - L_{ab} K'^b + L'_a{}^c L_{cb} - L_a{}^c L'_{cb} + R_{abcd} K^c K'^d \tag{25}$$

gives an isomorphism $\phi_U : \mathcal{K}_U \longrightarrow \mathcal{P}_U$ of Lie algebras:

$$\phi_U([K, K']) = [\phi_U(K), \phi_U(K')] \tag{26}$$

For $x \in U$, define the subspace $W_{U,x}$ of W_x by

$$W_{U,x} := \{w \in W_x : w = X(x), \ \text{ for some } \ X \in \mathcal{P}_U\} \tag{27}$$

Since parallel sections of a vector bundle are determined by their value at a single point, $\mathcal{P}_U$ and $W_{U,x}$ are isomorphic as vector spaces via the restriction map $r_x : \mathcal{P}_U \longrightarrow W_{U,x}$, given by $r_x(X) := X(x)$. Comparing (17) and (25), we have, in fact, a Lie algebra isomorphism $r_x : \mathcal{P}_U \longrightarrow \mathcal{A}(W_{U,x}, R_x, g_x)$. Composing this with ϕ_U characterizes the Lie algebra of $\mathcal{K}_U$.

Lemma 5

$$r_x \circ \phi_U : \mathcal{K}_U \longrightarrow \mathcal{A}(W_{U,x}, R_x, g_x) \tag{28}$$

is an isomorphism of Lie algebras.

In order to find the Lie algebra of the local Killing fields of g about the point x it remains to calculate $W_{U,x}$ for a sufficiently small neighbourhood U of x. This is accomplished in the following section.

Lemma 6 $\mathcal{A}(V^* \oplus \Lambda^2 V^*, B = 0, h)$ *is isomorphic to the Lie algebra of the semidirect product* $\Re^n \times_{sd} SO(p,q)$.

Proof:
Let $M = U = \Re^n$. Identify V and T_xM, for some fixed $x \in M$, by a vector space isomorphism $\theta : T_xM \longrightarrow V$. Set $g_x := \theta^*(h)$, the pull-back of h. g_x extends naturally to a flat metric g of signature (p,q) defined on all of M. The Lie algebra of Killing fields $\mathcal{K}$ is then isomorphic to the Lie algebra of $\Re^n \times_{sd} SO(p,q)$. Furthermore, $W_{M,x} = T_x^*M \oplus \Lambda^2 T_x^*M$ and as Lie algebras,

$$\mathcal{A}(V^* \oplus \Lambda^2 V^*, B = 0, h) \cong \mathcal{A}(T_x^*M \oplus \Lambda^2 T_x^*M, R_x = 0, g_x) \cong \mathcal{K}$$

where the second isomorphism follows from Lemma 5.
q.e.d.

This leads to the following global result.

Proposition 7 *If the Riemann curvature tensor vanishes at some point* $x \in M$ *then the Lie algebra of Killing fields is isomorphic to a subalgebra of the Lie algebra of* $\Re^n \times_{sd} SO(p,q)$.

Proof:
Suppose $R_x = 0$. For $U = M$, Lemma 5 gives the isomorphism $\mathcal{K} \cong \mathcal{A}(W_{M,x}, R_x = 0, g_x)$. This is a subalgebra of $\mathcal{A}(W_x, R_x = 0, g_x)$, which by Lemma 6 is isomorphic to the Lie algebra of $\Re^n \times_{sd} SO(p,q)$.
q.e.d.

4 Parallel Fields and Locally Symmetric Spaces

We begin by briefly reviewing the method from [1]. This describes an algebraic procedure for determining the number of independent local parallel sections of a smooth vector bundle $\pi : W \to M$ with a connection ∇. Since the existence theory is based upon the Frobenius Theorem, smooth data are required.

Let W' be a subset of W satisfying the following two properties:

P1: the fibre of W' over each $x \in M$ is a linear subspace of the fibre of W over x, and

P2: W' is *level* in the sense that each element $w \in W'$ is contained in the image of a local smooth section of W', defined in some neighbourhood of $\pi(w)$ in M.

Let X be a local section of W'. The covariant derivative of X is a local section of $W \otimes T^*M$. Define $\widetilde{\alpha}$ by

$$\widetilde{\alpha}(X) := \phi \circ \nabla(X)$$

where $\phi : W \otimes T^*M \to (W/W') \otimes T^*M$ denotes the natural projection taken fibrewise. If f is any differentiable function with the same domain as X then

$$\widetilde{\alpha}(fX) = f\widetilde{\alpha}(X)$$

This means that $\widetilde{\alpha}$ defines a map

$$\alpha_{W'} : W' \to (W/W') \otimes T^*M$$

which is linear on each fibre of W'.

The kernel of $\alpha_{W'}$ is a subset of W', which satisfies property P1 but not necessarily property P2. In order to carry out the above constructions to $ker\, \alpha_{W'}$, as we did to W', the non-level points in $ker\, \alpha_{W'}$ must be removed. To this end we define a leveling map $\mathcal{S}$ as follows. For any subset V of W satisfying P1 let $\mathcal{S}(V)$ be the subset of V consisting of all elements v for which there exists a smooth local section $s : U \subseteq M \to V \subseteq W$ such that $v = s(\pi(v))$. Then $\mathcal{S}(V)$ satisfies both P1 and P2.

We may now describe the construction of the maximal flat subset $\widetilde{W}$, of W. Let

$$\begin{aligned} V^{(0)} &:= \{w \in W \mid F(,)(w) = 0\} \\ W^{(i)} &:= \mathcal{S}(V^{(i)}) \\ V^{(i+1)} &:= ker\, \alpha_{W^{(i)}} \end{aligned}$$

where, as before, $F : TM \otimes TM \otimes W \to W$ is the curvature tensor of ∇. This gives a sequence

$$W \supseteq W^{(0)} \supseteq W^{(1)} \supseteq \cdots \supseteq W^{(k)} \supseteq \cdots$$

of subsets of W. For some $k \in N$, $W^{(l)} = W^{(k)}$ for all $l \geq k$. Define $\widetilde{W} = W^{(k)}$, with projection $\tilde{\pi} : \widetilde{W} \to M$.

We say that the connection ∇ is *regular at* $x \in M$ if there exists a neighbourhood U of x such that $\tilde{\pi}^{-1}(U) \subseteq \widetilde{W}$ is a vector bundle over U. $\widetilde{W}_x$ shall denote the fibre of $\widetilde{W}$ over $x \in M$.

Lemma 8 *Let ∇ be a connection on the smooth vector bundle $\pi : W \to M$.*
(i) If $X : U \subseteq M \to W$ is a local parallel section then the image of X lies in $\widetilde{W}$.
(ii) Suppose that ∇ is regular at $x \in M$. Then for every $w \in \widetilde{W}_x$ there exists a local parallel section $X : U \subseteq M \to \widetilde{W}$ with $X(x) = w$.

We may now describe the Lie algebra of local Killing fields about a point x.

Theorem 9 *Let g be a smooth metric on a manifold M with associated connection ∇ on $W = T^*M \oplus \Lambda^2 T^*M$, which is assumed to be regular at $x \in M$. Then g has $dim\, \widetilde{W}_x$ independent local Killing fields in a sufficiently small neighbourhood U of $x \in M$. Moreover, the Lie algebra of Killing fields on U is canonically isomorphic to the Lie algebra $\mathcal{A}(\widetilde{W}_x, R_x, g_x)$.*

Proof:
By Lemma 8 there exists a sufficiently small open neighbourhood U of x such that $W_{U,x} = \widetilde{W}_x$. The theorem now follows from Lemma 5.
q.e.d.

As an illustration, we provide a short algebraic proof of the classical theorem that locally symmetric spaces satisfy

$$R \star R = 0 \tag{29}$$

(cf. [7], pg. 197, 2nd and 3rd equations).

Lemma 10 *If M is locally symmetric then $T^*M \subseteq \widetilde{W}$.*

Proof:

Suppose M is locally symmetric and let $x \in M$. Then there is an open neighbourhood U of x such that the space of Killing fields on U, whose covariant derivative vanishes at x, has dimension n. By the isomorphism given in Lemma 5, $T^*_x M \subseteq W_{U,x}$. From Lemma 8 (i) we have $W_{U,x} \subseteq \widetilde{W}_x$ and so $T^*_x M \subseteq \widetilde{W}_x$.

q.e.d.

Let $T = T_{a\cdots}$ be an n-tensor with $n \geq 1$. Define $R \cdot T$ to be the $(n+2)$-tensor obtained by contracting the rightmost index of R with the leftmost index of T:

$$R \cdot T_{abc\cdots} := R_{abc}{}^{s} T_{s\cdots} \tag{30}$$

Let $\mathsf{p} := T^*M$, the cotangent space of M and let $\mathsf{p}^{(1)}$ be defined as the set of all elements $v \in T^*M$ satisfying

$$R \star R \cdot v = 0 \tag{31}$$

Define t to be the set of all $L \in \Lambda^2 T^*M$ such that

$$R \star L = 0 \tag{32}$$

We shall assume that t has constant rank.

Proof of (29):

Let us calculate the derived flag of W supposing M to be locally symmetric. From (8), $W^{(0)} = V^{(0)} := ker\, F = \mathsf{p} \oplus \mathsf{t}$. Let $X = K + L$ be a local section of $W^{(0)}$, where K and L are local sections of p and t, respectively. By definition, $X(x) \in V^{(1)}_x$ if and only if $\nabla_i X(x) \in W^{(0)}_x$, for all i. This is equivalent to

$$(*): \quad L_{ab;(i)} - R_{ab(i)}{}^c K_c \in \mathsf{t} \quad \text{at } x.$$

Taking the covariant derivative of $R \star L = 0$ gives $R \star L_{;(i)} = 0$. Thus $L_{ab;(i)}$ is a local section of t. This means that $(*)$ holds if and only if $R \cdot K_x$ lies in t. This is the case precisely when $K \in \mathsf{p}^{(1)}$ at x, and so $V^{(1)} = \mathsf{p}^{(1)} \oplus \mathsf{t}$. By Lemma 10, we must have $V^{(1)} = T^*M \oplus \mathsf{t}$, whence it follows that $\mathsf{p}^{(1)} = T^*M$ and $W^{(1)} = V^{(1)} = W^{(0)}$. Therefore $R \star R = 0$.

q.e.d.

We have also shown that for M locally symmetric, $\widetilde{W} = W^{(0)} = \mathsf{p} \oplus \mathsf{t}$. Conversely, if $\widetilde{W} = \mathsf{p} \oplus \mathsf{t}$ then $T^*M \subseteq \mathit{ker}\, F$, from which we conclude that M is locally symmetric. The terminal subbundle of the derived flag therefore characterizes locally symmetric spaces:

$$M \text{ is a locally symmetric space if and only if } \widetilde{W} = \mathsf{p} \oplus \mathsf{t}. \tag{33}$$

In particular, the derived flag computes the local canonical decomposition.

As observed above, a Riemannian manifold with $\mathit{dim}\ M \geq 3$ has constant curvature if and only if $R \star L = 0$ for all $L \in \Lambda^2 T^*M$ (Proposition 2). Furthermore, all manifolds of dimension $n = 1$ or 2 satisfy $R \star L = 0$ for $L \in \Lambda^2 T^*M$ (cf. Lemma 1). Since spaces of constant curvature are locally symmetric the curvature F vanishes for such manifolds. It is not difficult to see that the converse holds (for the 2-dimensional case use the canonical form of the Riemann curvature: $R_{ijkl} = c(g_{il}g_{jk} - g_{ik}g_{jl})$, where c is the Gaussian curvature). Consequently,

$$M \text{ is a space of constant curvature if and only if } \widetilde{W} = W. \tag{34}$$

Employing Theorem 9, we obtain as a corollary the familiar result that a Riemannian manifold possesses the maximal possible number $\frac{1}{2}n(n+1)$ of independent local Killing fields if and only if it is a space of constant curvature.

5 Classification for Riemannian Surfaces

As a final illustration of Theorem 9, we shall determine which Riemannian surfaces correspond to the various types of Lie algebra $\mathcal{K}_x$. In the process, a necessary condition for a Riemannian surface to possess a Killing field is obtained. First, we recall the situation involving the maximal number of local Killing fields:

Let M be a Riemannian surface. The following are equivalent:

(i) dim $\mathcal{K}_x = 3$ for all $x \in M$.

(ii) M has constant Gaussian curvature c.

(iii) M is locally symmetric.

In this case,

(a) if $c = 0$ then $\mathcal{K}_x$ is isomorphic to the Lie algebra of $\Re^2 \times_{sd} SO(2)$;

(b) if $c > 0$ then $\mathcal{K}_x \cong \mathsf{sl}_2\Re$;

(c) if $c < 0$ then $\mathcal{K}_x \cong \mathsf{su}_2$.

The equivalence of (i)-(iii) follows from Lemma 1, (34) and the observation below (34). Assume that these conditions hold. $c = 0$ corresponds, locally, to flat Euclidean space, for which $\mathcal{K}_x$ is isomorphic to the Lie algebra of the semidirect product of translations and rotations. Suppose $c \neq 0$ and let X and Y be orthogonal vectors in T_x^*M with norm

$$X^2 = Y^2 = \frac{1}{|c|}$$

Define $H \in \Lambda^2 T_x^* M$ by

$$[X, Y] := H$$

Then

$$[H, X] = -sg(c)Y \qquad \text{and} \qquad [H, Y] = sg(c)X$$

where $sg(c)$ denotes the sign of c. Appealing to Lemma 5, this identifies $\mathcal{K}_x$ with $\mathsf{sl}_2\Re$ for $c > 0$ and with su_2 for $c < 0$ (cf. [5], pg. 143).

Next, we calculate the derived flag for W assuming that the surface is *regular*: $W^{(i)} = V^{(i)}$ has constant rank for all i. The elements $K \in T^*M$ satisfying $R_{ijkl}{}^{;s}K_s$

are those for which $c^{;s}K_s = 0$. Therefore, by Lemma 1,

$$W^{(0)} = ker\,\partial c \oplus \Lambda^2 T^* M \tag{35}$$

where ∂c denotes the vector field $c^{;a}$. The case $\partial c = 0$ has been handled above. Suppose therefore that $dim\,ker\,\partial c = 1$; that is, ∂c is non-vanishing. Then $W^{(0)}$ is a rank two fibre bundle over M. Let $X = K + L$ be a local section of $W^{(0)}$, where K is a local section of $ker\,\partial c$ and L is a local section of $\Lambda^2 T^* M$. $X(x) \in W_x^{(1)}$ is equivalent to $\nabla_i X(x) \in W_x^{(0)}$ for all i, by the definition of the derived flag. Since $\Lambda^2 T^* M \subseteq W^{(0)}$, $X(x) \in W_x^{(1)}$ if and only if

$$K_{a;(i)} - L_{a(i)} \in ker\,\partial c \tag{36}$$

at x. Taking the covariant derivative of $c^{;s}K_s = 0$ gives the equation $c^{;a}K_{a;b} = -c_{;ab}K^a$. Substituting this into (36) determines $W^{(1)}$ as the subset of all $K + L \in W^{(0)}$ satisfying

$$c_{;ab}K^a + L_{ab}c^{;a} = 0 \tag{37}$$

By contracting (37) with K and ∂c, it is evident that $W^{(1)}$ consists of the zero elements in W along with the solutions of

$$c_{;ab}K^a K^b + L_{ab}c^{;a}K^b = 0 \tag{38}$$

$$c_{;ab}K^a c^{;b} = 0 \tag{39}$$

where $0 \neq K \in ker\,\partial c$ and $L \in \Lambda^2 T^* M$. Equation (39) has a solution $0 \neq K \in ker\,\partial c$ if and only if

$$c_{;ab}c^{;b} = f c_a \tag{40}$$

for some f. (40) may be written in terms of differential forms as

$$dc \wedge D_{\partial c} dc = 0 \tag{41}$$

where D denotes covariant differentiation. This leads to a necessary condition for the existence of a Killing field on a Riemannian surface.

Theorem 11 *If a regular Riemannian surface possesses a Killing field then*

$$dc \wedge D_{\partial c} dc = 0$$

Proof:

By Theorem 9, if a regular Riemannian surface has a Killing field then $\widetilde{W}$ has rank at least one. Since $\widetilde{W} \subseteq W^{(1)}$, equation (39) must have a non-trivial solution $K \in ker\, \partial c$ at each $x \in M$. The theorem now follows from the fact that (39) is equivalent to (41).

q.e.d.

Corollary 12 *Let M be a regular Riemannian surface with non-constant curvature. If M possesses a Killing field then the integral curves of ∂c are geodesic paths.*

By a *geodesic path* we mean a curve that is a geodesic when appropriately parameterized.

Proof:

Equation (41) is equivalent to $D_{\partial c}\partial c = f\partial c$, which implies that integral curves of the non-vanishing vector field ∂c may be parametrized so as to give geodesics of M.

q.e.d.

An example would be the punctured paraboloid $z = x^2 + y^2$; $(x, y) \neq (0, 0)$, with the induced metric from its embedding into 3-dimensional Euclidean space. The integral curves of ∂c are described by the geodesic paths $\gamma_\theta(t) = (tcos\theta, tsin\theta, t^2)$, up to reparametrization.

Now let us return to calculating $W^{(1)}$. If $dc \wedge D_{\partial c} dc = 0$ on the surface then the non-zero elements of $W^{(1)}$ are the solutions to (38) for which $0 \neq K \in ker\, \partial c$. For any choice of non-trivial $K \in ker\, \partial c$, (38) uniquely determines an element $L = L(K) \in \Lambda^2 T^* M$. Therefore $W^{(1)}$, in this case, is a rank one vector bundle over M. If, on the other hand, $dc \wedge D_{\partial c} dc \neq 0$ on M then $W^{(1)}$ is the zero bundle and there do not exist any local Killing fields. As a consequence, $\mathcal{K}_x$ cannot be 2-dimensional;

this may also be seen directly by considering the Lie bracket operation. Henceforth we shall assume that $dc \wedge D_{\partial c} dc = 0$.

To find $W^{(2)}$, let $X = K+L$ be a local section of $W^{(1)}$. By definition, $X(x) \in W_x^{(2)}$ if and only if $\nabla_i X(x) \in W_x^{(1)}$ for all i. Owing to (37), this is equivalent to

$$c_{;ab} K'^{a}{}_{(i)} + L'_{ab(i)} c^{;a} = 0 \tag{42}$$

where

$$\begin{aligned} K'_{a(i)} &:= K_{a;(i)} - L_{a(i)} \\ L'_{ab(i)} &:= L_{ab;(i)} - R_{ab(i)}{}^{c} K_c \end{aligned}$$

(Note that from the description of $W^{(1)}$ contained in (36) it follows that $K'_{a(i)} + L'_{ab(i)} \in W^{(0)}$.) Taking the covariant derivative of (37) gives

$$c_{;ab} K^{a}{}_{;(i)} + L_{ab;(i)} c^{;a} = -c_{;ab(i)} K^{a} - L_{ab} c^{;a}{}_{(i)} \tag{43}$$

Substituting (43) into (42) defines $W^{(2)}$ as the subset of all $K + L \in W^{(1)}$ such that

$$c_{;abc} K^{a} + L_{ab} c^{;a}{}_{c} + L_{ac} c^{;a}{}_{b} + R_{abcd} c^{;a} K^{d} = 0 \tag{44}$$

If this has only the trivial solution then $\widetilde{W} = W^{(2)}$ is the zero bundle. Otherwise, $\widetilde{W} = W^{(2)} = W^{(1)}$ has rank one.

Theorem 13 *Let M be a regular Riemannian surface. Then $dim\, \mathcal{K}_x = 1$ for all $x \in M$ if and only if*
(i) $dc \neq 0$,
(ii) $dc \wedge D_{\partial c} dc = 0$, *and*
(iii) equation (44) holds for all $K + L \in W^{(1)}$.

Proof:
Conditions (i)-(iii) are equivalent to $rank\, \widetilde{W} = 1$. The result now follows from Theorem 9.
q.e.d.

We summarize the discussion in this section with the following corollary.

Corollary 14 *For regular Riemannian surfaces, $\mathcal{K}_x$ may be one of five possible Lie algebras. It is isomorphic to either $\mathsf{sl}_2\Re$, su_2 or the Lie algebra of $\Re^2 \times_{sd} SO(2)$ when the Gaussian curvature is constant and positive, negative or zero, respectively. $\mathcal{K}_x$ is the 1-dimensional Lie algebra when the conditions of Theorem 13 are met. Otherwise, there do not exist any local Killing fields and $\mathcal{K}_x$ is trivial.*

References

[1] Atkins, R.: On Parallel Sections of a Vector Bundle. arXiv 0804-1732

[2] Atkins, R.: When is a Connection a Levi-Civita Connection? arXiv 0804-2698

[3] Berger, M.: Trois remarques sur les variétés riemaniennes à courboure positive. C. R. Acad. Sci. Paris Sér. A-B 263, 76-78 (1966)

[4] Console, S., Olmos ,C.: Curvature Invariants, Killing Vector Fields, Connections and Cohomogeneity. arXiv 0801-3211v2

[5] Fulton, W., Harris, J.: Representation Theory. Graduate Texts in Mathematics, Volume 129. Springer, 1991

[6] Gorbatsevich, V.V., Onishchik, A.L.: Lie groups of transformations. Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya 20, 103-240 (1988)

[7] Helgason, S.: Differential Geometry, Lie Groups, and Symmetric Spaces. Graduate Studies in Mathematics, Volume 34. Am. Math. Soc., 2001

[8] Killing, W.: Über die Grundlagen der Geometrie. J. reine angew Math. 109, 121-186 (1892)

[9] Kostant, B.: Holonomy and the Lie algebra of infinitesimal motions of a Riemannian manifold. Trans. Am. Math. Soc. 80 (1955)

[10] Tano, S. Killing vectors and geodesic flow vectors on tangent bundles. J. reine angew. Math. 282, 162-171 (1976)

[11] Vinberg, E.B.: Geometry II. Encyclopaedia of Mathematical Sciences, Volume 29. Springer-Verlag, 1993

[12] Wolf, J.A.: Spaces of Constant Curvature. Berkeley: University of California, 1972